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Note

Separable Subsets of a Finite Lattice

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One of the basic themes of combinatorial lattice theory is concerned with the role of the irreducible elements of a lattice. The importance of these elements stems from the elementary fact that every element of a finite lattice L is both the join and the meet of irreducible elements of L. Equivalently, for elements a and b of L, $a \nleqslant b$, there exists a (join) irreducible element a' such that $a' \leqslant a$ and $a' \nleqslant b$ and there exists a (meet) irreducible element b' such that $b' \geqslant b$ and $b' \ngeqslant a$. Loosely speaking, distinct elements of L can be "separated" by the irreducible elements of L.

For a lattice L let $\mathbf{J}(L)$, respectively $\mathbf{M}(L)$, denote the partially ordered subset of join irreducible, respectively meet irreducible, elements of L and let $\mathbf{P}(L) = \mathbf{J}(L) \cup \mathbf{M}(L)$ denote the partially ordered subset of irreducible elements of L. If a and b are noncomparable elements of a finite lattice L then there exists $a' \in \mathbf{P}(L)$ such that $a' \leqslant a$ and $a' \nleqslant b$ whence, $\{a, b\} \cong \{a', b\}$ as partially ordered subsets of L. On the other hand, if $L \cong \mathbf{3} \times \mathbf{3}$ is the direct product of two three-element chains and a, b, c are pairwise noncomparable elements of L then at least one of these elements is not irreducible, say a (see Fig. 1), and for every $a' \in \mathbf{P}(L)$, $\{a, b, c\} \ncong \{a', b, c\}$ as partially ordered subsets of L; that is, $\{a, b, c\}$ cannot be "separated" by the irreducible elements of L.

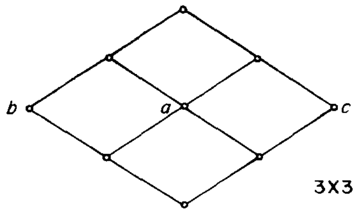

FIGURE 1

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For partially ordered sets P and Q a mapping φ of P to Q is a *weak embedding* if φ is one-to-one and both φ and φ^{-1} are order-preserving. For a partially ordered set P and a lattice L we say that P is *separable in L* if for every weak embedding φ of P to L and for every $a \in \varphi(P)$ there is $a' \in \mathbf{P}(L)$ such that, as partially ordered subsets of L,

$$\varphi(P) \cong \{a'\} \cup (\varphi(P) - \{a\}).$$

For instance, we have already remarked that a two-element antichain is separable in any finite lattice, while a three-element antichain is not even separable in $\mathbf{3} \times \mathbf{3}$. The purpose of this paper is to characterize the finite partially ordered sets which are separable in every finite lattice.

For $n = 1, 2, \ldots$ let $\mathbf{A}_n$ denote the partially ordered set in Fig. 2. (Notice

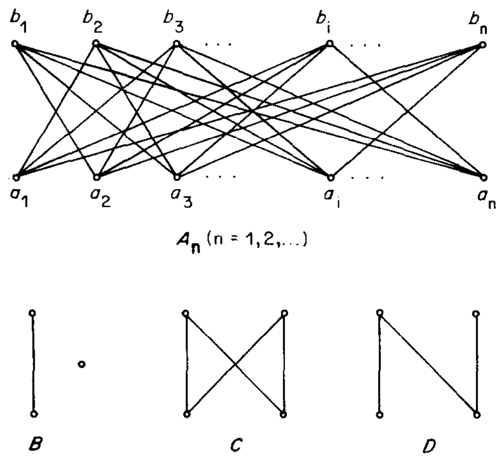

FIGURE 2

that $\mathbf{A}_1$ is a two-element antichain, $\mathbf{A}_2$ is the disjoint union of two two-element chains and, for $n = 3, 4, \ldots$, $\mathbf{A}_n$ is isomorphic to the set of one-element and $(n-1)$-element subsets of $\{1, 2, \ldots, n\}$ partially ordered by set inclusion.) Let us suppose that $\mathbf{A}_n$, labeled as in Fig. 2, is contained in a finite lattice L. Let $a \in \mathbf{A}_n$. If $a = a_i$ we choose a join irreducible element a' such that $a' \leqslant a$ and $a' \nleqslant b_i$; if $a = b_i$ we choose a meet irreducible element a' such that $a' \geqslant a$ and $a' \ngeqslant a_i$. In any case, $\mathbf{A}_n \cong \{a'\} \cup (\mathbf{A}_n - \{a\})$; whence, $\mathbf{A}_n$ is separable in every finite lattice. Similarly, each of the partially ordered sets $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ (see Fig. 2) is also separable in every finite lattice. The substance of this paper lies in proving that these are the only finite partially ordered sets separable in every finite lattice.

THEOREM. *For a finite partially ordered set P the following conditions are equivalent*:

(1) *P is separable in every finite lattice*;

(2) *if P is weakly embeddable in a finite lattice L then P is weakly embeddable in* $\mathbf{P}(L)$;

(3) *P is isomorphic to* $\mathbf{A}_n$ $(n = 1, 2,...)$, $\mathbf{B}$, $\mathbf{C}$, *or* $\mathbf{D}$.

That (1) implies (2) is obvious from the definition of separability, while (3) implies (1) is a consequence of the observations above. Before we proceed to the proof that (2) implies (3) we dispense with several preliminary remarks.

Let L be a finite lattice and, for each $a \in L$, set $J(a) = \{x \in \mathbf{J}(L) \mid x \leqslant a\}$. Then $L' = \{J(a) \mid a \in L\}$ partially ordered by $\subseteq$ is a lattice isomorphic to L; moreover, L' is weakly embeddable in the lattice $\mathbf{2}^m$, $m = |\mathbf{J}(L)|$, of all subsets of $\{1, 2,..., m\}$ partially ordered by $\subseteq$.

LEMMA 1. *Every finite lattice is weakly embeddable in the lattice of all subsets of a finite set.*

Let P be a partially ordered set and, for each $S \subseteq P$, set $S^* = \{x \in P \mid x \geqslant s \text{ for every } s \in S\}$ and $S_* = \{x \in P \mid x \leqslant s \text{ for every } s \in S\}$. The mapping ν defined by $\nu(S) = (S^*)_*$ is a closure operator on P and the mapping $a \to \nu(a)$ is a weak embedding of P into the complete lattice $\mathbf{L}(P) = \{\nu(S) \mid S \subseteq P\}$ partially ordered by set inclusion. In fact, the weak embedding ν of P into $\mathbf{L}(P)$ preserves all existing joins and meets of P. $\mathbf{L}(P)$ is the well-known *normal completion* of P introduced by MacNeille [3] (cf. [2]) as a generalization to arbitrary partially ordered sets of the familiar construction of the real numbers from the rationals by "Dedekind cuts."

Let $S \in \mathbf{L}(P)$. Then $S = \bigvee(\nu(a) \mid a \in S)$. Indeed, $\nu(a) = \{x \in P \mid x \leqslant a\}$ so that $S \subseteq \bigcup (\nu(a) \mid a \in S)$. Since $S = (S^*)_*$, $\nu(a) \subseteq S$ for each $a \in S$; whence, $S = \bigcup (\nu(a) \mid a \in S) = \bigvee(\nu(a) \mid a \in S)$. Furthermore, since $\mathbf{L}(P) \simeq (\mathbf{L}(P^d)^d)$ (K^d denotes the dual of K) we also have that $S = \bigwedge(\nu(a) \mid a \in T)$ for some $T \subseteq P$. Therefore, if we identify P with its image under ν in $\mathbf{L}(P)$ then every element of $\mathbf{L}(P)$ is both a join and a meet of subsets of P. On the other hand, $\mathbf{L}(P)$ is the unique (up to isomorphism) complete lattice L containing P in which every element is both a join and a meet of elements of P (cf. [1, 4]).

We call an element a of a partially ordered set P *doubly reducible* if there exist $A, B \subseteq P - \{a\}$ such that $\sup(A) = a = \inf(B)$. Let $\mathbf{R}(P)$ denote the subset of all doubly reducible elements of P. Notice that $\sup(A) = \bigvee A$ $(\inf(A) = \bigwedge A)$ in $\mathbf{L}(P)$ if $\sup(A)$ $(\inf(A))$ exists for $A \subseteq P$.

The next result concerning the role of doubly reducibles in $\mathbf{P}(\mathbf{L}(P))$ provides a useful technique in our proof of the Theorem. The result is due to Kelly and Rival (unpublished).

LEMMA 2. *For any finite partially ordered set* P

$$\mathbf{P}(\mathbf{L}(P)) \cong P - \mathbf{R}(P).$$

Proof. Let Q be a partially ordered set with no doubly reducible elements. Let $a \in Q$ and let us suppose that $\nu(a) \notin \mathbf{P}(\mathbf{L}(Q))$, say $\nu(a) = \bigvee(\nu(x) \mid x \in S)$ for some $S \subseteq Q$, where $a \notin S$. Then $a = \sup(x \mid x \in S)$. Hence, $\mathbf{R}(Q) = \varnothing$ implies that Q is weakly embeddable in $\mathbf{P}(\mathbf{L}(Q))$ so $Q \cong \mathbf{P}(\mathbf{L}(Q))$.

Finally, if P is an arbitrary finite partially ordered set then $\mathbf{R}(P - \mathbf{R}(P)) = \varnothing$; whence $P - \mathbf{R}(P) \cong \mathbf{P}(\mathbf{L}(P - \mathbf{R}(P))) \cong \mathbf{P}(\mathbf{L}(P))$. ∎

We call a partially ordered set P *disconnected* if P is the disjoint union of subsets A and B such that for each $a \in A$ and for each $b \in B$, a is noncomparable to b; otherwise, P is *connected.* Let $w(P)$ denote the *width* of P, that is, the size of a maximum-sized antichain in P.

We are ready to complete the proof of the Theorem.

Proof of Theorem. Let P be a finite partially ordered set satisfying (2). In view of Lemma 1, $\mathbf{L}(P)$ is weakly embeddable in $\mathbf{2}^m$ for some m, whence, $P \subseteq \mathbf{P}(\mathbf{2}^m)$. It follows that $P = \min(P) \cup \max(P)$ where $\min(P)$ denotes the minimal elements of P and $\max(P)$ denotes the maximal elements of P. Moreover, for each $a \in \min(P)$ ($a \in \max(P)$) there is at most one $b \in \max(P)$ ($b \in \min(P)$) such that $a \nleqslant b$ ($b \nleqslant a$). (Notice that $\mathbf{P}(\mathbf{2}^m) \cong \mathbf{A}_m$ for $m \neq 2$.)

Let P be disconnected. Then, either $P \cong \mathbf{B}$, $P \cong \mathbf{A}_2$ or P is an antichain. Hence, let us assume that P is an antichain. As any antichain is weakly embeddable into the direct product of two sufficiently long chains C_1, C_2 and $w(P(C_1 \times C_2)) = 2$ we conclude that $|P| = 2$; that is, $P \cong \mathbf{A}_1$. (Notice that a singleton is weakly embeddable in $\mathbf{L}(\varnothing)$ and $\mathbf{P}(\mathbf{L}(\varnothing)) = \varnothing$.)

Let P be connected. We set $A = \{x \in \min(P) \mid x < y \text{ for every } y \in \max(P)\}$ and $B = \min(P) - A$. We distinguish cases according to the sizes of A and B.

Case (i). Let $|A| \geqslant 3$. Let a_1, a_2, a_3 be distinct elements of A and let c_1, c_2, c_3 be distinct elements disjoint from P. We consider $P' = P \cup \{c_1, c_2, c_3\}$ with the partial ordering induced by P and the comparabilities: $a_1 < c_1$, $c_2 < a_1$, $c_3 < a_1$, $c_2 < a_2$, and $c_3 < a_3$. Let $b \in P$ such that $b > a_1$. Then $\sup(\{c_2, c_3\}) = a_1 = \inf(\{b, c_1\})$; in fact $\mathbf{R}(P') = \{a_1\}$. Applying Lemma 2 we have that

$$\mathbf{P}(\mathbf{L}(P')) \cong P' - \{a_1\} = (P - \{a_1\}) \cup \{c_1, c_2, c_3\}.$$

Since P is weakly embeddable in $\mathbf{L}(P')$ we must have that P is weakly embeddable in $P' - \{a_1\}$. This, however, could occur only if

$$w((\min(P) - \{a_1\}) \cup \{c_2, c_3\}) \geqslant |\min(P)|.$$

Since

$$w((\min(P) - \{a_1\}) \cup \{c_2, c_3\}) = |\min(P)| - 1$$

we conclude that $|A| \leqslant 2$.

Case (ii). Let $|A| = 2$. If $B = \varnothing$ let C_1, C_2 be sufficiently long chains and let L be the *ordinal sum* of the lattices $\mathbf{2}^2$ and $C_1 \times C_2$ ($x < y$ for each $x \in \mathbf{2}^2$ and for each $y \in C_1 \times C_2$). Since P is weakly embeddable in L and $w(\mathbf{P}(L)) = 2$ we conclude that $|\max(P)| \leqslant 2$. If $|\max(P)| = 2$ then $P \cong \mathbf{C}$. If $|\max(P)| = 1$ then P is weakly embeddable in $\mathbf{2}^2$ although P is not weakly embeddable in $\mathbf{P}(\mathbf{2}^2) \cong \mathbf{A}_1$.

If $B \neq \varnothing$, let $A = \{a_1, a_2\}$ and choose $a_3 \in B$. We may construct P' as in case (i); again, P is weakly embeddable in $\mathbf{L}(P')$ but P is not weakly embeddable in $\mathbf{P}(\mathbf{L}(P')) = P' - \{a_1\}$.

Case (iii). Let $A = \{a\}$. If $B = \varnothing$ and $n = |\max(P)|$ then P is weakly embeddable in the $(n + 2)$-element lattice L in which every chain has three elements. Since $\mathbf{P}(L)$ is an antichain P is not weakly embeddable in $\mathbf{P}(L)$.

Let $B = \{b\}$. As P is connected, $|\max(P)| \geqslant 2$. If $|\max(P)| = 2$ then $P \cong \mathbf{D}$. Otherwise, let d_1, d_2, d_3 be distinct elements of $\max(P)$ satisfying the comparabilities: $d_1 > a$, $d_1 \ngeqslant b$, $d_2 > a$, $d_2 > b$, $d_3 > a$, and $d_3 > b$. Let c_1, c_2, c_3 be distinct elements disjoint from P and consider $P' = P \cup \{c_1, c_2, c_3\}$ with the partial ordering induced by P and the comparabilities: $c_1 < d_3$, $d_3 < c_2$, $d_3 < c_3$, $d_1 < c_2$, and $d_2 < c_3$. Then $\sup(\{a, c_1\}) = d_3 = \inf(\{c_2, c_3\})$ and $\mathbf{R}(P') = \{d_3\}$. In view of Lemma 2

$$\mathbf{P}(\mathbf{L}(P')) \cong (P - \{d_3\}) \cup \{c_1, c_2, c_3\}.$$

Since

$$w((\max(P) - \{d_3\}) \cup \{c_2, c_3\}) < |\max(P)|$$

P cannot be weakly embeddable in $\mathbf{P}(\mathbf{L}(P')) \cong P' - \{d_3\}$.

Let b_1, b_2 be distinct elements of B. Let c_1, c_2 be distinct elements disjoint from P and consider $P' = P \cup \{c_1, c_2\}$ with the partial ordering induced by P and the comparabilities: $c_1 < a$, $c_1 < b_1$, $c_2 < a$, and $c_2 < b_2$. Again, P is weakly embeddable in $\mathbf{L}(P')$, yet P is not weakly embeddable in

$$\mathbf{P}(\mathbf{L}(P')) \cong (P - \{a\}) \cup \{c_1, c_2\}.$$

Case (iv). Let $A = \varnothing$. By duality we may suppose that for every $b \in \max(P)$ there is precisely one $b' \in \min(P)$ such that $b \ngeqslant b'$. It follows at once that $|\min(P)| = |\max(P)|$; whence $P \cong \mathbf{A}_n$ for some $n \geqslant 3$. ∎

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